JOURNAL OF APPROXIMATION THEORY 44, 69–72 (1985)

In a Semi-infinite Program Only a Countable Subset of the Constraints Is Essential

D. F. KARNEY

School of Business, University of Kansas, Lawrence, Kansas 66045, U.S.A.

Communicated by Oved Shisha

Received March 14, 1984

In this note, it is shown that, for an arbitrary semi-infinite convex program, there exists a countable subcollection of the constraints which gives the primal program and whose dual gives the original dual value.

1. INTRODUCTION

Whenever we find ourselves confronted with a semi-infinite program, it is reasonable to want to immediately say that, without loss of generality, we may assume that the number of constraints is countable. As Example 2.2 illustrates, this reduction is not as easy as selecting a countable number of the constraints which gives the feasible region of the primal program. The problem lies with the dual value. Here we show that there always exists a countable subcollection of the constraints which gives an equivalent primal program and yields the correct dual value. We conclude by citing conditions under which every countable subcollection of the constraints, which gives an equivalent primal program, gives the same dual value.

2. THE RESULT

Consider the semi-infinite convex program $\mathscr{P}$ which seeks

$$(\mathscr{P}) \qquad \mathscr{M}\mathscr{P} = \inf f(x) \quad \text{s.t.}\, g_i(x) \leqslant 0; \qquad i \in I$$

0021-9045/85 $3.00

and its formal Lagranian dual $\mathscr{D}$ which seeks

$$(\mathscr{D})\qquad \begin{aligned} \mathscr{M}\mathscr{D} = \sup_{\lambda} \inf_{x} L(x,\ \lambda) \\ \text{s.t. } \lambda \geqslant 0 \\ \lambda \in \Lambda. \end{aligned}$$

In the above formulation, $x \in \mathbb{R}^p$, the functions f and g_i for all $i \in I$ are closed proper convex, the cardinality of the index set I is infinite,

$$\Lambda = \{(\lambda_i)_{i \in I} \mid \lambda_i \neq 0 \text{ for only finitely many } i\}$$

and

$$L(x, \lambda) = f(x) + \sum_{i \in I} \lambda_i g_i(x).$$

It now follows directly, from the separability of $\mathbb{R}^p$, and the fact that each closed proper convex function is the supremum of its affine minorants, that there exists a countable subset $I_0 \subset I$ satisfying

$$\{x \mid g_i(x) \leqslant 0; i \in I_0\} = \{x \mid g_i(x) \leqslant 0; i \in I\}. \tag{2.1}$$

In other words, the program $\mathscr{P}(I_0)$ which seeks

$$(\mathscr{P}(I_0))\qquad \begin{aligned} \mathscr{M}\mathscr{P}(I_0) = \inf f(x) \\ \text{s.t. } g_i(x) \leqslant 0; \qquad i \in I_0 \end{aligned}$$

is equivalent to $\mathscr{P}$. As the next example illustrates, different choices of I_0 satisfying (2.1) can result in different values for $\mathscr{M}\mathscr{D}(I_0)$.

EXAMPLE 2.2.

$$\begin{aligned} &\inf \exp(-y) \\ &\text{subject to } (x^2 + y^2)^{1/2} - x \leqslant 0 \\ &\qquad x \geqslant \gamma \qquad \text{for all } \gamma < 0. \\ &\qquad y \leqslant \mu \qquad \text{for all } \mu > 0. \end{aligned}$$

Let I_1 be those indices of I which correspond to

$$\begin{aligned} &(x^2 + y^2)^{1/2} - x \leqslant 0 \\ &\quad x \geqslant -1/i; \qquad i = 2, 3, \ldots \end{aligned}$$

and let I_2 be those indices of I which correspond to

$$(x^2+y^2)^{1/2}-x\leqslant 0$$

$$y\leqslant\frac{1}{i};\qquad i=2, 3,\ldots.$$

Since $(x^2+y^2)^{1/2}-x\leqslant 0$ implies $y=0$ and $x\geqslant 0$, both I_1 and I_2 satisfy (2.1). However, $\mathcal{MD}(I_1)=0$ while $\mathcal{MD}(I_2)=1$. The first statement follows from the fact that $\inf_{x,y} L(x, y, \lambda)>-\infty$ if and only if $\lambda=(\lambda_1, 0, 0,\ldots)$. The second from the facts that, for $n\geqslant 2$, $\inf_{x,y} L(x, y, \lambda^n)=\exp(-1/n)$, where $\lambda^n=(0,\ldots 0, \lambda_n=\exp(-1/n), 0\ldots)$ and that $\mathcal{MP}(I_2)=1$. (For more on $\mathcal{P}(I_2)$ see [2, 5].)

Note, in the above example, that $\mathcal{MD}(I_2)=\mathcal{MD}$. As seen in the next theorem, this is not an accident.

THEOREM 2.3. *Given a semi-infinite program $\mathcal{P}$ with the number of constraints, I, uncountable, there exists a countable subset I_0 of I with the following properties*:

$$\{x\mid g_i(x)\leqslant 0; i\in I_0\}=\{x\mid g_i(x)\leqslant 0; i\in I\} \tag{2.1}$$

$$\mathcal{MP}(I_0)=\mathcal{MP} \tag{2.4}$$

$$\mathcal{MD}(I_0)=\mathcal{MD}. \tag{2.5}$$

Proof. Let I_0 be a countable subset of I satisfying (2.1) and (2.4). If $\mathcal{MD}=-\infty$, then (2.5) also holds for I_0 since $\mathcal{MD}(I_0)\leqslant\mathcal{MD}$.

Now assume that $\mathcal{MD}$ is finite and let k be a fixed positive integer. Then there exists a λ^k feasible for $\mathcal{D}$ with $\inf_x L(x, \lambda^k)\geqslant\mathcal{MD}-k^{-1}$. With this λ^k, construct a countable subset I_k of I by adding to I_{k-1} those i in $I\backslash I_{k-1}$ for which $\lambda_i^k\neq 0$. Let $J=\bigcup_{k=0}^{\infty} I_k$. Then J is countable since each I_k is and J satisfies (2.1) and (2.4) since each I_k does. Lastly, $\mathcal{MD}(J)=\mathcal{MD}$ since for $k\geqslant 1$, $\mathcal{MD}(I_k)\geqslant\mathcal{MD}-k^{-1}$ by construction.

Finally, if $\mathcal{MD}=+\infty$, repeat the above argument with k in place of $\mathcal{MD}-k^{-1}$. ∎

DEFINITION. A countable subset I_0 of I is *essential* iff it satisfies properties (2.1), (2.4) and (2.5).

The conclusion of the above theorem can now be sharpened in the case that $\mathcal{P}$ satisfies a general condition which implies $\mathcal{MP}=\mathcal{MD}$, for instance, when the feasible region of $\mathcal{P}$ is nonempty and bounded (for this condition, see [4]. For other conditions, see [1, 3, 6]). In this case, all subsets I_0 satisfying (2.1) are essential. This follows directly from the fact that $\mathcal{MP}(I_0) = \mathcal{MD}(I_0)$ for all such I_0.

In [6], this last observation is utilized to derive a general duality result for the finite subprograms of $\mathcal{P}$ associated with each I_0.

References

1. J. M. Borwein, The limiting Lagrangian as a consequence of Helly's theorem, *J. Optim. Theory Appl.* **33** (1981), 479–513.
2. R. J. Duffin, Convex analysis treated by linear programming, *Math. Programming* **4** (1973), 125–143.
3. R. G. Jeroslow, Uniform duality in semi-infinite convex optimization, *Math. Programming* **27** (1983), 144–154.
4. D. F. Karney, Clark's theorem for semi-infinite convex programs, *Adv. in Appl. Math.* **2** (1981), 7–12.
5. D. F. Karney, A pathological semi-infinite program verifying Karlovitz's conjecture, *J. Optim. Theory Appl.* **38** (1982), 137–141.
6. D. F. Karney, A duality theorem for semi-infinite convex programs and their finite subprograms, *Math. Programming* **27** (1983), 75–82.